# Optimal Design Method for Selective Nerve Stimulation and Its Application to Electrocutaneous Display

Hiroyuki Kajimoto  Naoki Kawakami  Susumu Tachi

School of Information Science and Technology
The University of Tokyo
Bunkyo-ku Hongo 7-3-1, Japan
{kaji,kawakami,tachi}@star.t.u-tokyo.ac.jp

## Abstract

*We have developed a tactile display that uses electric current from the skin surface as a stimulus. Our main objective was to independently stimulate a variety of mechanoreceptors and to generate specific sensations by combining stimuli. The key to this goal is selective nerve stimulation. In this paper, a mathematical framework is build for the general design of the selective stimulation. The geometries of electrodes and nerve fibers are arbitrary, and the waveform of the electric current from each electrode is independently controlled. Furthermore, the problem is formulated by linear or quadratic programming, which provides an optimal solution.*

## 1 Introduction

An electrocutaneous display is a tactile device that directly activates nerve fivers within the skin by electric current from surface electrodes.

Electrocutaneous displays are superior to conventional mechanical tactile displays in many respects. They are smaller, more durable, more energy-efficient, and free from many mechanical difficulties such as resonance. For these reasons, extensive work has been done on electrocutaneous displays [6, 7].

One of the most important subject of electrocutaneous display is selective nerve stimulation. There are two aspects to it. One is to avoid invasive sensation during stimulation, which is achieved by suppressing pain related nerves while activating the nerves of a mechanoreceptor.

The other is to independently stimulate each type of mechanoreceptor. If thiese steps are achieved, any kind of tactile sensation could be generated by combining specific stimuli [2]. Although it is a more difficult task, we have already shown [7] that selective stimulation of each type of mechanoreceptor is possible. We call these stimuli "tactile primary colors" because they are analogous to red, green, and blue, which are the primary colors for vision.

This design challenge is not limited to the electrocutaneous display. It is common to any study of electrical stimulation, and, hence, there have been many studies on the subject [4, 5, 10, 11]. However, although some of them analyzed phenomena with precise modeling and some were based on physiological experiments, few of them dealt with the design of optimal stimulation using arbitrary nerve geometry. Perhaps the concept of *Activating Function* [11] was a rare successful case, but it only dealt with geometry and does not provide insights into waveforms.

In this paper, a mathematical framework is presented for a general design of selective stimulation. The geometries of electrodes and nerve fibers are arbitrary, and the waveforms of the electric current from each electrode are independently controlled. Furthermore, the selective stimulation problem is formulated by linear or quadratic programming, which provides an optimal solution.

## 2 Electrical Models

In this section, a mathematical model, which describes the spatial and temporal relationship between electrodes and nerve fibers, is formulated.

Figure 1 is an illustration of a general situation of electrical stimulation from the skin surface and an electrical model of nerve fibers [9]. For simplicity, this paper deals exclusively with a 2D cross-sectional problem, but it may be expanded easily to 3D.

We assume that a nerve is activated when the po-

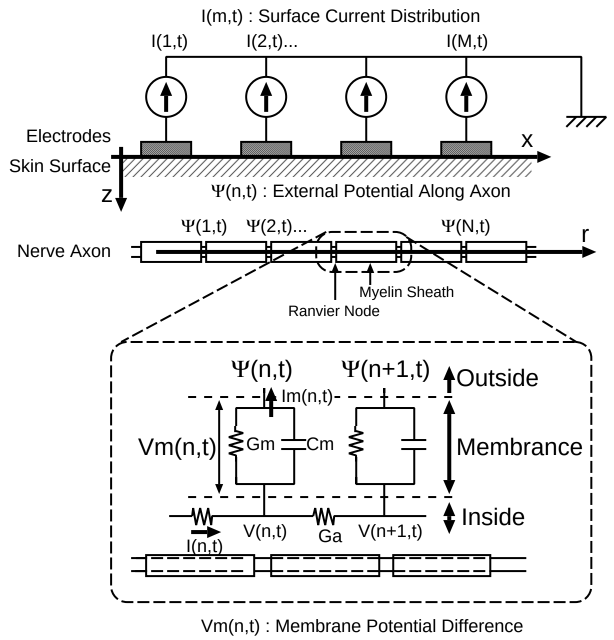


**Figure 1. Current Stimulation from the Skin Surface and an Electrical Model of a Nerve Fiber.**

tential difference of nerve membrane $V_m$ reaches a certain threshold $V_{th}$. Then, electrical stimulation could be recognized as a mapping problem between a current source distribution $I(x,t)$ on the skin surface and $V_m(r,t)$, where $x$ and $r$ are coordinates along the skin surface and nerve fiber and $t$ represents the time.

This mapping $I(x,t) \to V_m(r,t)$ is divided into two processes. The first process is described as a mapping between $I(x,t)$ and the external membrane potential $\Psi(r,t)$ along a nerve. The second process is a mapping between the external membrane potential $\Psi(r,t)$ and the membrane potential difference $V_m(r,t)$.

## 2.1 Mapping 1: $\Psi(r,t) \to V_m(r,t)$

The second mapping $\Psi(r,t) \to V_m(r,t)$ will be dealt with first. Here, the myelinated nerve, which is characterized by a surrounding insulator called the Myelin Sheath, is discussed. As almost all the nerve membrane is insulated by the sheath, it can only be electrically accessed by the Node of Ranvier, which is a small gap between each sheath. Since only these gaps need to be considered, a discrete model could be built as follows.

Numbers are given to these nodes. Then, the external membrane potential, membrane potential difference, and internal potential at these nodes are represented as $\Psi(n,t)$, $V_m(n,t)$, and $V(n,t)$, where $n(1 \leq n \leq N)$ is the node number.

The nerve membrane at the node is electrically modeled with capacitance $C_m(n)$ and conductance $G_m(n)$. The internal conductance from node $n-1$ to node $n$ is represented as $G_a(n)$. The electric current from the inside to the outside of the membrane at the node is $I_m(n,t)$, and the internal current from node $n-1$ to node $n$ is $I(n,t)$. When they are obvious, $n$ and $t$ will be omitted from now on.

From Kirchhoff's law of current,the membrane current density $I_m(n,t)$ must be equal to the loss of internal current $I$ at node $n$, and $I_m$ is also the sum of the current flowing $C_m$ and $G_m$.

$$\begin{aligned} & I_m(n) \\ & \quad = \quad I(n) - I(n+1) \\ & \quad = \quad G_a(V(n+1) - 2V(n) + V(n-1)) \quad (1) \\ & \quad = \quad C_m \frac{\partial V_m(n)}{\partial t} + G_m V_m(n) \quad (2) \end{aligned}$$

Putting $V_m + \Psi$ in place of $V$,

$$\begin{aligned} \frac{\partial V_m(n)}{\partial t} \quad = \quad & (-\frac{G_m}{C_m} - 2\frac{G_a}{C_m})V_m(n) \\ & + \frac{G_a}{C_m}(V_m(n+1) + V_m(n-1)) \\ & + \frac{G_a}{C_m}(\Psi(n+1) - 2\Psi(n) + \Psi(n-1)) \end{aligned}$$

The equation is simplified by the vector representation as follows [3]:

$$\dot{\mathbf{V}}_m = \mathbf{A}\mathbf{V}_m + \mathbf{B}\mathbf{\Psi} \quad (3)$$

$$\begin{aligned} \mathbf{V}_m &= [V_m(1), V_m(2), ..., V_m(N)]^{\mathrm{T}} \\ \mathbf{\Psi} &= [\Psi(1), \Psi(2), ..., \Psi(N)]^{\mathrm{T}} \\ \mathbf{A} &= \begin{bmatrix} -\frac{2G_a}{C_m} - \frac{G_m}{C_m} & \frac{G_a}{C_m} & 0 & 0 \\ \frac{G_a}{C_m} & -\frac{2G_a}{C_m} - \frac{G_m}{C_m} & \frac{G_a}{C_m} & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \frac{G_a}{C_m} & -\frac{2G_a}{C_m} - \frac{G_m}{C_m} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} -\frac{2G_a}{C_m} & \frac{G_a}{C_m} & 0 & 0 \\ \frac{G_a}{C_m} & -\frac{2G_a}{C_m} & \frac{G_a}{C_m} & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \frac{G_a}{C_m} & -\frac{2G_a}{C_m} \end{bmatrix}, \end{aligned}$$

where $\dot{}$ represents the temporal differential and the superscript T means the transverse vector.

## 2.2 Mapping 2: $I(x,t) \to \Psi(n,t)$

Next, another mapping $I(x,t) \to \Psi(n,t)$ is treated. Here, it is assumed that each electrode is small enough to be represented as a point source. Then, it could also be discretized as $I(m,t)$, where $m(1 \leq m \leq M)$ is the number of electrodes.

For simplicity, the uniform infinite space with pure resistance is considered. The electric potential $\Psi$ is obtained by an integral of the current density as follows:

$$\begin{aligned}\Psi(n,t) &= -\int_{R(n,1)} i\rho dR \qquad (4)\\ &= \frac{I(1,t)\rho}{4\pi R(n,1)^2}, \qquad (5)\end{aligned}$$

where $i$ is the current density, $\rho$ is the resistivity, and $R(n,1)$ is the distance between electrode 1 and node $n$.

For a case of arrayed electrodes, $\Psi(n,t)$ is obtained by superposition.

$$\Psi(n,t) = \frac{I(1,t)\rho}{4\pi R(n,1)^2} + \frac{I(2,t)\rho}{4\pi R(n,2)^2} + \cdots + \frac{I(M,t)\rho}{4\pi R(n,M)^2} \qquad (6)$$

The equation is simplified by the vector representation as follows:

$$\mathbf{\Psi}(t) = \mathbf{C}\mathbf{I}(t), \qquad (7)$$

where

$$\mathbf{I}(t) = [I(1,t), I(2,t), ..., I(M,t)]^{\mathrm{T}} \qquad (8)$$

$$\mathbf{C} = \frac{\rho}{4\pi}\begin{bmatrix} \frac{1}{R^2(1,1)} & \frac{1}{R^2(1,2)} & \cdots & \frac{1}{R^2(1,M)} \\ \frac{1}{R^2(2,1)} & \frac{1}{R^2(2,2)} & \ddots & \frac{1}{R^2(2,M)} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{R^2(N,1)} & \frac{1}{R^2(N,2)} & \cdots & \frac{1}{R^2(N,M)} \end{bmatrix} \qquad (9)$$

It has long been a matter of controversy whether current control is superior to voltage control in an electrical stimulation [5], and the discussions were mainly focused on the stability of sensation and the ease of circuit fabrication. However, a current source is much superior to a voltage source from a mathematical viewpoint. Since current sources were used, a generated electric potential field was easily calculated by a simple superposition of each current source. If a voltage source is used, the calculation of potential field is much more difficult.

From Eqs.3 and 7, the following is obtained:

$$\dot{\mathbf{V}}_m = \mathbf{A}\mathbf{V}_m + \mathbf{BCI} \qquad (10)$$

Finally, $\mathbf{x}$ for $\mathbf{V_m}$, $\mathbf{B}$ for $\mathbf{BC}$, $\mathbf{u}$ for $\mathbf{I}$ is used, and a system state equation is obtained.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (11)$$

## 3 Selective Stimulation

In this section, the optimal design method of selective nerve stimulation is described. The purpose of selective stimulation is simple. It is to stimulate the desired nerve fibers without stimulating undesired fibers. In this paper, a case with two nerve fibers, $a$ and $b$, will be discussed, in which fiber $a$ should be activated and fiber $b$ should be suppressed (Figure 2).

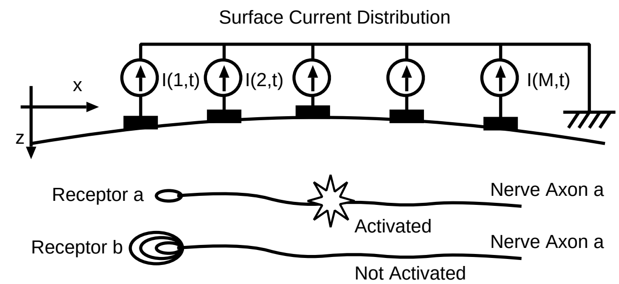


**Figure 2. Selective Nerve Stimulation by Electrocutaneous Stimulation.**

It is mathematically described as follows:

Let the system state equations of these two nerve fibers be described as follows:

$$\dot{\mathbf{x_a}} = \mathbf{A_a}\mathbf{x_a} + \mathbf{B_a}\mathbf{u} \qquad (12)$$

$$\dot{\mathbf{x_b}} = \mathbf{A_b}\mathbf{x_b} + \mathbf{B_b}\mathbf{u} \qquad (13)$$

The input vector $\mathbf{u}(t)$ is common in these equations. Our objective is to find $\mathbf{u}(t)$, which activates nerve $a$ while suppressing nerve $b$. Because of the assumption that the nerve will fire when the membrane potential difference reaches the threshold, the selective stimulation is described as an optimization problem as follows:

$$\max(\mathbf{x_b} \mid \max(\mathbf{x_a}) = V_{th}) \underset{\mathbf{u}}{\rightarrow} \min \qquad (14)$$

Its meaning is as follows: the maximum value of $V_m$ of nerve $b$ is minimized, while $V_m$ of nerve $a$ reaches the threshold voltage ($V_{th}$). Equations 12 through 14 are a Mini-Max problem of dynamic systems. The maximum value is searched both spatially and temporally.

### 3.1 Temporal Discretization

What makes the problem difficult in Eq.14 is that it is a dynamical system. In other words, although the spatial discretization has been completed, the temporal axis is still continuous. Here, the temporal region is discretized by assuming a pulsed waveform as follows:

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}(1) & 0 \le t < dT \\ \mathbf{u}(2) & dT \le t < 2dT \\ \vdots & \vdots \\ \mathbf{u}(T) & T_L - dT \le t < T_L \end{cases}$$

where $dT$ is the pulse width, $T_L$ is the stimulation period, and $T = T_L/dT$ is the number of pulses. Then, Eq.11 becomes a simple difference equation as follows:

$$\mathbf{x}(0) = \mathbf{0} \quad (15)$$

$$\mathbf{x}(k) = \mathbf{P}\mathbf{x}(k-1) + \mathbf{Q}\mathbf{u}(k) \quad (1 \le k \le T), \quad (16)$$

where $\mathbf{P} = \exp(\mathbf{A}dT)$ and $\mathbf{Q} = \mathbf{A}^{-1}(\mathbf{P} - \mathbf{I})\mathbf{B}$.

By lining up the vectors, the following is obtained:

$$\begin{bmatrix} \mathbf{x}(1) \\ \mathbf{x}(2) \\ \vdots \\ \mathbf{x}(T) \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ \mathbf{PQ} & \mathbf{Q} & 0 & 0 \\ \vdots & \mathbf{PQ} & \mathbf{Q} & 0 \\ \mathbf{P}^{T-1}\mathbf{Q} & \mathbf{P}^{T-2}\mathbf{Q} & \cdots & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{u}(1) \\ \mathbf{u}(2) \\ \vdots \\ \mathbf{u}(T) \end{bmatrix}$$

The equation is rewritten as follows:

$$\mathbf{x} = \mathbf{A}\mathbf{u}, \quad (17)$$

where $\mathbf{x}$, $\mathbf{A}$, and $\mathbf{u}$ are all redefined. $\mathbf{x}$ is the $NT \times 1$ vector, which contains both temporal and spatial information of the membrane potential difference, $\mathbf{u}$ is a $MT \times 1$ vector, and $A$ is a $NT \times MT$ matrix. Then, Eqs.12 and 13 are changed to

$$\mathbf{x_a} = \mathbf{A_a}\mathbf{u} \quad (18)$$

$$\mathbf{x_b} = \mathbf{A_b}\mathbf{u} \quad (19)$$

### 3.2 Safety Condition

In electrical stimulation, when electric current is applied from one electrode, the current should be returned by the surrounding electrodes. In other words, the sum of the current source distribution should always be zero at any moment. If this condition is not satisfied, the current flows deeply into the human body, which may cause unexpected serious trouble. Therefore, this is considered to be a safety condition. It is described as follows:

$$\sum_{m=1}^{M} I(m,t) = 0$$

or, simply,

$$\mathbf{E}\mathbf{u} = \mathbf{0}, \quad (20)$$

where $\mathbf{E}$ is $T \times MT$ matrix as follows:

$$\mathbf{E} = \begin{bmatrix} 1\cdots1 & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & 1\cdots1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0\cdots0 \\ 0\cdots0 & \cdots & 0\cdots0 & 1\cdots1 \end{bmatrix}$$

### 3.3 Formulation of the Optimization Problem as Linear Programming

From Eqs.14,18,19 and 20, the selective stimulation problem is rewritten as follows:

$$\max(\mathbf{A_b}\mathbf{u}) \underset{\mathbf{u}}{\to} \min \quad (21)$$

$$\text{subject to}$$

$$\max \mathbf{A_a}\mathbf{u} = V_{th} \quad (22)$$

$$\mathbf{E}\mathbf{u} = \mathbf{0} \quad (23)$$

This is a problem of linear inequality called MiniMax. Still, it is not easy to solve because there is an unusual condition $\max \mathbf{A_a}\mathbf{u} = V_{th}$ in the constraint equations. This equation means that the membrane potential difference of nerve $a$ reaches the threshold at *some* place and at *some* time.

The restriction is changed into an easier condition as follows: $V_m$ of nerve $a$ reaches $V_{th}$ at a *known* time and a *known* place. Let the place and the time be $n_{act}$ and $T$ (final time of stimulation), respectively. Then, Eq.22 is changed to

$$\mathbf{A_{a\_part}}\mathbf{u} = V_{th},$$

where $\mathbf{A_{a\_part}}$ is a partial matrix (actually a vector) from $\mathbf{A_a}$, which satisfies $\mathbf{A_{a\_part}}\mathbf{u} = x_a(n_{act}, T)$.

The final formula is as follows:

$$\max(\mathbf{A_b}\mathbf{u}) \underset{\mathbf{u}}{\to} \min \quad (24)$$

$$\text{subject to}$$

$$\mathbf{A_{a\_part}}\mathbf{u} = V_{th} \quad (25)$$

$$\mathbf{E}\mathbf{u} = \mathbf{0} \quad (26)$$

By a simple transformation (which is described in the Appendix), it is changed to a standard form of linear programming. After that, it is numerically solved by mathematical software, such as Matlab™.

## 4 Other Constraints

In this section, some other practical constraints of electrical stimulation are discussed, and the method for building them into our optimization formula is described.

### 4.1 Balanced or Biphasic Condition

Although this is not the main focus of our study, it is worth mentioning that some researchers suggest that balanced stimulation is necessary for long-term (1 minute∼) stimulation, in which positive and negative part of waveform is equalized. It is described as follows:

$$\int_{t=0}^{T_L} I(m,t)dt = 0 \tag{27}$$

or, simply,

$$\mathbf{Fu} = \mathbf{0} \tag{28}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \end{bmatrix}$$

where $\mathbf{F}$ is the $M \times MT$ matrix and $\mathbf{I}$ is a $M \times M$ identity matrix. This condition is quite similar to our safety condition in Sec.3.2.

### 4.2 Minimization of Joule Heat

One problem of electrical stimulation is that a sudden sensation of stinging occurs because a concentrated current produces Joule heat, which activates untargeted afferent nerves [8].

The solution to this problem is to find an optimal waveform in which stimulation can be achieved with a minimum of Joule heat.

In general, the calculation of Joule heat ($J$) is not easy. We regard it as proportional to the square of the electric current. Although it is not accurate, we expect that the results may give us some insight for the stimulation design.

$$J \propto \sum_{m=1}^{M} \int_{t=0}^{T} I(m,t)^2 dt \tag{29}$$

$$\propto \mathbf{u}^{\mathrm{T}} \cdot \mathbf{u} \tag{30}$$

Therefore, a mathematical representation of the minimization of Joule heat is

$$\mathbf{u}^{\mathrm{T}} \cdot \mathbf{u} \underset{\mathbf{u}}{\rightarrow} \min \tag{31}$$

### 4.3 Electric Current Limitation

The other problem is the limitation of the stimulator circuit because there is no such thing as an infinitely high-voltage stimulator. Ordinarily, the limitation appears as the maximum and minimum electric current. It is represented as follows:

$$\mathbf{1} \cdot I_{\min} \leq \mathbf{u} \leq \mathbf{1} \cdot I_{\max} \tag{32}$$

Our optimization formula, which satisfies Eqs.28 and 32 and optimizes Eq.31 as well as Eqs.24 through 26. is as follows:

$$\mathbf{u}^{\mathrm{T}}\mathbf{u} + w \max(\mathbf{A_b u}) \underset{\mathbf{u}}{\rightarrow} \min \tag{33}$$

subject to

$$\mathbf{A_{a\_part}u} = V_{th} \tag{34}$$

$$\mathbf{Eu} = \mathbf{0} \tag{35}$$

$$\mathbf{Fu} = \mathbf{0} \tag{36}$$

$$\mathbf{1} \cdot I_{\max} \geq \mathbf{u} \geq \mathbf{1} \cdot I_{\min} \tag{37}$$

where $w$ is a weight parameter. It is arbitrary chosen by the designer of the stimulation. If $w = 0$, only Joule heat is optimally minimized, while if $w \rightarrow \infty$, it is equivalent to the previous linear programming. The mathematical framework was changed from linear to quadratic programming.

## 5 Optimal Design Examples

In this section, the design for the optimal stimulation in some simple situations is described. By comparing our results with previously obtained knowledge, we show the relevance of our method.

In all examples, the electrical parameters of a nerve were from [9]. The nerve was horizontally oriented to the skin surface with 20[mm] (20-40 Nodes of Ranvier) in length, and stimulation length $T_L$ was fixed to 200[$\mu$s]. A Node of Ranvier just beneath the central electrode is chosen as a node to be activated ($n_{act}$).

### 5.1 Minimum Energy Stimulation

The first example is a design of a waveform to stimulate a single nerve while minimizing generated Joule heat. As the Joule heat is a main cause of pain during stimulation, the obtained waveform is expected to realize the stimulation with a higher pain threshold. This problem is formulated when we put $w = 0$ in Eq.33. Equations 36 through 37 were not used.

The situation is illustrated in Figure 3. We placed three electrodes 1[mm] apart. As the total current density is always zero (safety condition from Sec.3.2), and from the geometrical symmetry, the waveform from the central electrode must be considered. The electric current of the surrounding two electrodes is $-1/2$ of that.

The unit sampling time $dT$ as 10[$\mu$s] was set, and the optimal waveform from the central electrode was calculated. The result is shown in Figure 4. The cathodic (depolarizing) current is taken in a positive direction.

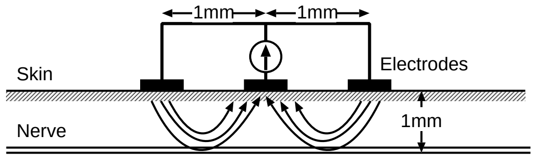


**Figure 3. Situation of the Minimum Energy Stimulation.**

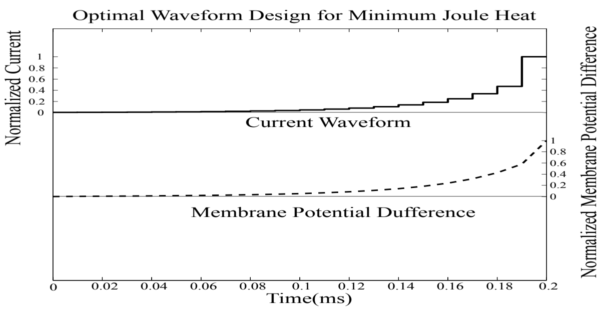


**Figure 4. Optimal Waveform and Membrane Potential of the Central Node in the Minimum Energy Stimulation.**

The membrane potential difference of the central node is also shown.

The obtained optimal solution was a cathodic, exponentially increasing waveform. The total Joule heat was decreased by approximately 30% compared to a rectangular wave, whose pulse width is optimized.

This waveform could be roughly regarded as a combination of two pulses, one of which is a low-level, depolarizing pre-pulse, and the other is a main short pulse. From this viewpoint, our result may explain the result obtained by Poletto [10], who experimentally obtained a higher pain threshold by a low-level, long-term depolarizing pre-pulse.

### 5.2 Diameter Selective Stimulation

The second example is a waveform design for diameter selective stimulation. It is useful for electrocutaneous display because different types of cutaneous information are transmitted by nerves of different diameters.

There is a well-known relationship between pulse width and nerve diameter. A thicker nerve is selectively stimulated if a pulse is short, while a thinner nerve is also stimulated if a pulse becomes wider. We expected the same tendency to be observed.

We used Eqs.21 through 23 to obtain an optimal waveform. Most of the situation is the same as in the previous example, including the unit sampling time $dT$. However, there are two nerve fibers with different diameters ($5[\mu$m] and $10[\mu$m]) at the same depth.

The results of thicker and thinner nerve stimulation are shown in Figure 5.

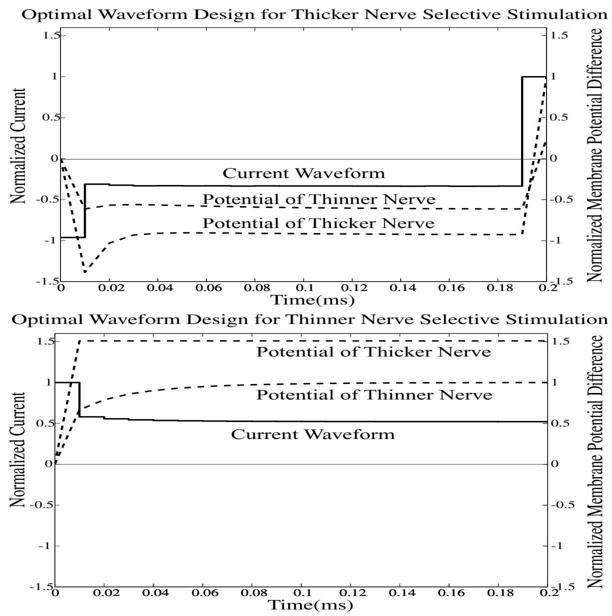


**Figure 5. Optimal Waveform and Membrane Potential. (Top): For Thicker Nerve. (Bottom): For Thinner Nerve.**

It was obvious that the thicker nerve selective stimulation had the shortest pulse width possible, while the thinner nerve stimulation had the longest pulse width, which agreed well with our expectations.

There are two minor but interesting features. One is that, in thicker nerve stimulation, a long-term, low level anodic current (hyperpolarization) was observed. The other is that, in thinner nerve stimulation, a high, short-term cathodic (depolarization) input was seen at the first period. This characteristic agreed well with the result obtained by Fang [4]. He has shown experimentally that a Quasi-Trapezoidal pulse that was amazingly similar in shape to our optimized waveform could selectively activate small nerve fibers. This close agreement implied that both Fang's experimental result and our own mathematically obtained result concering optimization are relevant.

### 5.3 Depth Selective Stimulation

The previous two examples dealt with waveform only. Here, we deal with how to design the spatial patterns of electrodes. The example treats spatial weight optimization for depth selective stimulation in which shallower or deeper nerves are selectively activated.

The situation is illustrated in Figure 6. Seven electrodes were placed 1[mm] apart on the skin surface. Two nerves with the same diameter were located at a depth of 1[mm] and 2[mm], respectively.

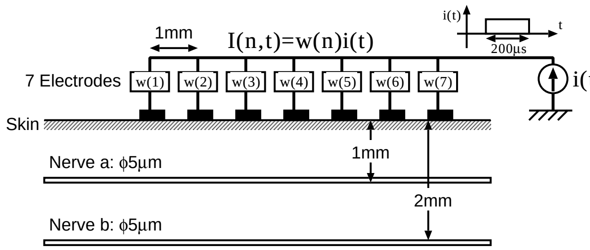


**Figure 6. Situation of Depth Selective Stimulation.**

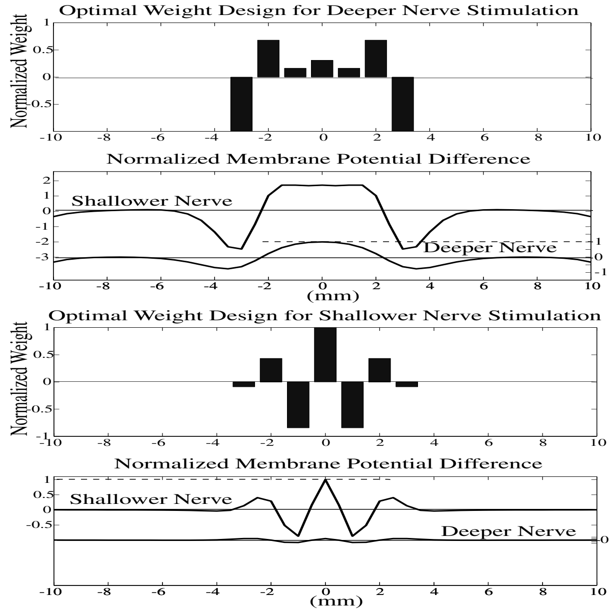


**Figure 7. Optimal Weights and Membrane Potential. (Top): For Deeper Nerve. (Bottom): For Shallower Nerve.**

Equations 21 through 23 were used again. The stimulation waveform was fixed at a 200[$\mu$s] rectangular pulse. It was achieved by setting $dT = T_L$ (or T=1).

There are also well-known facts about a coaxial electrode[1]. If the diameter of a central electrode is enlarged, or if the distance between the central electrode to the surrounding electrode is enlarged, stimulation of deeper tissues is possible; whereas, if it is not, they could only stimulate shallower tissues. We expected the same tendency to be observed. The results are shown in Figure 7.

As we expected, deeper nerve stimulation had a wide cathodic weights at the center, which enables electric current to reach deep inside the skin. On the contrary, a shallower nerve stimulation was achieved by a pair of central cathodic and surrounding anodic current. This pair worked as an electrical dipole, which enabled fast attenuation of electric potential.

These results agreed well with our previous results [7], which were designed with *Activating Function* [11].

Until now, we only used multiple electrodes and obtained optimal weights for them. Although this is not discussed in depth in the present study, we could group some cathodic and anodic electrodes and represent them with larger electrodes. If we did that, we could say that we had designed the electrode shape.

[1]Electrodes composed of a central stimulating electrode and a surrounding return electrode

## 6 Conclusion

We constructed a mathematical framework for the design of selective stimulation. The selective stimulation was described as a sort of Mini-Max problem of dynamic systems, in which the membrane potential difference was a state vector and the electric current was an input vector.

We then discretized the system both spatially and temporally and showed that the problem could be understood as linear or quadratic programming, which automatically gives us an optimal solution.

When arbitrary geometries of electrodes and nerve fibers are given, optimal waveforms from each electrode are automatically calculated, as we have seen in Sections.5.1 and 5.2. The design of an optimal electrode shape is also possible, as we have shown in Sec.5.3.

In Sec.5, we saw that the optimal solutions we obtained agreed well with previous studies. This agreement indicates the following three significant ideas. One is, of course, the relevance of our method, regardless of our very rough modeling and many assumptions. Another is the usefulness, which is understood by the fact that previous results were obtained with great efforts of trial and error, while we obtained them by a single mathematical optimization. The last is that we could assure that previously proposed stimuli were *optimal* in a mathematical sense.

Although we dealt with some of the simplest cases in this paper, we do not think it would be difficult to expand our method to an actual case. Our next step will be to find an optimal stimulation method for the selective stimulation of mechanoreceptors in a finger.

**Acknowledgements** This work was partly supported by the JSPS Research Fellowships for Young Scientists.

## Appendix: Mini-Max Problem Solution

We have seen in Sec.3.3 that our optimization problem is reduced to linear inequality, a Mini-Max problem. To numerically solve it, we must transform the problem into a standard form of linear programming, which the mathematical software can manage. Here, we take "linprog"[2] as an example and show the transformation.

The "linprog" solves a standard linear programming problem as follows:

$$\mathbf{f}^{\mathrm{T}}\mathbf{x} \underset{\mathbf{x}}{\rightarrow} \min \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{38}$$

$$\mathbf{A_{eq}}\mathbf{x} = \mathbf{b_{eq}} \tag{39}$$

$$\mathbf{lb} \leq \mathbf{x} \leq \mathbf{ub}, \tag{40}$$

where $\mathbf{f}$, $\mathbf{x}$, $\mathbf{b}$, $\mathbf{b_{eq}}$, $\mathbf{lb}$, and $\mathbf{ub}$ are vectors and $\mathbf{A}$ and $\mathbf{A_{eq}}$ are matrices [1]. We now transform Eqs.24 through 26 to this canonical form.

Let the maximum value of $\mathbf{A_b u}$ be $b_{\max}$. Then,

$$\mathbf{A_b u} \quad \leq \quad [1, 1, \cdots, 1]^{\mathrm{T}} b_{\max} \tag{41}$$

Adding $b_{\max}$ at the end of $\mathbf{u}$, $\mathbf{x}$ is defined as follows:

$$\mathbf{x} = [\mathbf{u}^{\mathrm{T}}, b_{\max}]^{\mathrm{T}}$$

Note that

$$b_{\max} = \mathbf{f}^{\mathrm{T}}\mathbf{x} \tag{42}$$

$$\mathbf{f}^{\mathrm{T}} \quad = \quad [0, 0, \cdots, 0, 1]$$

Then, Eq.41 is expressed as

$$\mathbf{A}\mathbf{x} \leq 0 \tag{43}$$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{A_b} & -\mathbf{1} \end{bmatrix} \\ -\mathbf{1} &= \begin{bmatrix} -1 & -1 & \cdots & -1 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

[2] *linprog* is a function of Matlab™

From Eqs.42 and 43, the final representation of our optimization problem is

$$\mathbf{f}^{\mathrm{T}}\mathbf{x} \underset{\mathbf{x}}{\rightarrow} \min \quad \text{subject to} \tag{44}$$

$$\mathbf{A}\mathbf{x} \leq 0 \tag{45}$$

$$\mathbf{A_{eq}}\mathbf{x} = \mathbf{b_{eq}}, \tag{46}$$

where

$$\begin{aligned} \mathbf{A_{eq}} &= \begin{bmatrix} \mathbf{A_{a\_part}} & \mathbf{0} \\ \mathbf{E} & \mathbf{0} \end{bmatrix} \\ \mathbf{b_{eq}} &= \begin{bmatrix} V_{th} & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

This formula is in the scope of the "linprog", so that we could solve it with Matlab.

Here, we dealt with linear programming, but things do not change in the case of quadratic programming, which we discussed in Sec.4.

## References


[1] *Optimization Toolbox in Matlab.* Mathworks Inc., http://www.mathworks.com/.

[2] N. Asamura, N. Yokoyama, and H. Shinoda. A method of selective stimulation to epidermal skin receptors for realistic touch feedback. In *Proc. IEEE VR'99*, pages 274–281, 1999.

[3] B. Coburn. A theoretical study of epidural electrical stiumuation of the spinal cord - part ii: Effects on lnog myelinated fibers. *IEEE Trans. Biomed. Eng.*, 32(11):978–986, Nov 1985.

[4] Z.-P. Fang and J. T. Mortimer. Selective activation of small motor axons by quasitrapezoidal current pulses. *IEEE Trans. Biomed. Eng.*, 38(2):168–174, Feb 1991.

[5] W. M. Grill and J. T. Mortimer. Stimulus waveforms for selective neural stimulation. *IEEE Eng. in Med. & Biol.*, 14:375–385, 1995.

[6] K. A. Kaczmarek, M. E. Tyler, and P. B. y Rita. Electrotactile haptic display on the fingertips: Preliminary results. In *Proc. 16th Annu. Int. Conf. IEEE Eng. Med. Biol. Soc.*, pages 940–941, 1994.

[7] H. Kajimoto, N. Kawakami, T. Maeda, and S. Tachi. Tactile feeling display using functional electrical stimulation. In *Proc. of The 9th Int. Conf. on Artificial reality and Telexistence*, pages 107–114, Dec 1999 http://www.ic-at.org.

[8] J. L. Mason and N. A. M. Mackay. Pain sensations associated with electrocutaneous stimulation. *IEEE Trans. Biomed. Eng.*, BME-23(5):405–409, Sep 1976.

[9] D. R. McNeal. Analysis of a model for excitation of myelinated nerve. *IEEE Trans. Biomed. Eng*, BME-23(4):329–337, Jul 1976.

[10] C. J. Poletto. Fingertip electrocutaneous stimulaiton through small electrodes. *Doctoral Thesis, Case Western Reserve University*, Jan 2001.

[11] F. Rattay. Modeling axon membranes for functional electrical stimulation. *IEEE Trans. Biomed. Eng.*, 40(12):1201–1209, Dec 1993.